Algebraic Number Theory

Exercise Sheet 10

Prof. Dr. Nikita Geldhauser
PD Dr. Maksim Zhykhovich

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Exercise 1. Let d be a square-free integer, $d \not\equiv 1 \bmod 4$. Let $K = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = d$. Let p be a prime number in $\mathbb{Z}$. Describe the decomposition of the ideal $(p) \in \mathfrak{F}(\mathcal{O}_K)$ into a product of prime ideals.
Hint: Use Satz 42 (see Exercise 1, Sheet 7) from the lecture.

Exercise 2. Let $K = \mathbb{Q}(\alpha)$ be a quadratic field, where $\alpha^2 = -6$. Find $C(\mathcal{O}_K)$ for $d = -6$.
Hint for (1) and (2): Use the inequality from Korollar 9 (Chapter II) and estimate $\left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|d_K|}$.

Exercise 3. Let A be a Dedekind ring. Show that every ideal $I \subset A$ can be generated by two elements.
Hint: Take $a \in I \setminus \{0\}$. Decompose the ideals I and (a) into a product of prime ideals and compare these decompositions. Use Exercise 1, Sheet 9, to find $x \in A$, such that $(a, x) = I$.